The alternation of representations in case of Cantor set

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ABSTRACT: It has been advocated that the concept of infinity does not allow to be pierced by intuition. In this article we present a comparison between of two kinds of the concept infinity countable and uncountable. Employing varieties representations of the Cantor set and the step function we unfold in the intuition the comparison of these two cardinalities. We attempt an application of the theory of representations in a concrete issue of undergraduate studies demonstrating the necessity of the combination of both the iconic and symbolic representations therefore bringing in the action intuitive as well manipulate skills for a global and critical understanding.

Introduction: In mathematics, every discussion is committed to a system of symbols, a model into which is constructed the meaning. When we discuss about representations in mathematics education, we use the concept in order to distinguish between the different semiotic systems (Vergnaud, 1998) by which we can express the “same” object. In this point we have to do an observation. When we have to deal with a real problem there is legitimate talk about representations of the same object. Talking the classical Frege's distinction between sense and reference there are two signs "morning star" and "evening star" that they have different senses but the same virtue of their reference to the planet Venus. Have we the same distinction when we work in two different representational systems (Goldin, 1998) in mathematics? The answer is no. In mathematics the reference is not a real object but a mental object constructed into the representational system itself.

For example on the foundation of analytic geometry we have imposed an isomorphism between geometric space $\mathbb{R}^n$, with $n=1, 2, 3$, the vector space $\mathbb{R}^n$, $n= 1,2, 3$ and the vector space $\mathbb{R}^n$ of (of $n$-tuples, for $n =1,2,3$) that they called $n$-dimensional point space, translation space, and co-ordinate space, respectively, (Jaeger, 1964).

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Therefore when we treat the conic sections. There is a meaning of the ellipse in the classical geometry. It is defined as:

A conic section is a curve obtainable on the surface of a circular cone at the intersection with an arbitrary plane P that does not pass through the vertex of the cone. The case of ellipse, we can roughly say, is if the angle of the plane P and the base of the cone is less than the slope of the cone.

This is a geometric meaning and beyond this meaning in this concrete geometric model there is not any other object of the reality. By another representation we can have the ellipse in linear representation in the form of equation or determinant. It is a new concept and its meaning belongs in the semiotic context of linear algebra. The two forms with the same name are not representations of the same thing. They are two isomorphic mental objects into two different isomorphic, (Greer & Harel, 1998) representational systems.

The main concept, i.e. the ellipse, works as a kind of prototype: an idea that there is some core part of meaning that is invariant across all contexts or instances (Malt, 1999, p 333). Such different expressions, as the representations, are very useful in education in order to help a global function of the brain, an integration of different competence of the mind. A classification provided by Lawer is interesting. He argues about psychological value of representations:

Some people report that their thoughts are very rich in imagery (visual thinkers) others that thinking proceeds as an internal dialogue (audile thinkers), and for others that internal kinesethetics is paramount (motile thinkers) … However, if the different sensory motor systems encode their memories and respond to external stimuli in terms of different primitives, it would not be strange if human’s different representations also reflect the origins of the specific experience through which the knowledge was acquired. I proposed that representations for use by people must related to possibilities of the human system.

Lawer (1996, p. 247)

He makes a distinction joined to mathematical competence as: the visually oriented minds favorable to Venn diagrams, language oriented to logical and algebraic thinking, manipulation oriented for constructive geometric proofs.

It is classical Bruner’s distinction (1988) between iconic and symbolic. As a modification of this idea in case of mathematical notions is the idea presented by Tall & Vinner (1981), Vinner (1983, 1992) about the visualisation of a concept has to do with it. Our visual intuition has a primitive representational value for our thought, but the general iconic is an arbitrary attribution to intuition and we should make clear what we
mean. Kant, very early, does a distinction between the image and the schema and gives implicitly the concept of representation:

If five points be set alongside one another, thus, ..., I have an image of the number five. But if, on the other hand, I think only a number in general, whether it be five or a hundred, this thought is rather the representation of a method whereby a multiplicity, for instance a thousand, may be represented in an image in conformity with a certain concept, than the image itself. For with such a number as a thousand the image can hardly be surveyed and compared with the concept. This representation of a universal procedure of imagination in providing an image for a concept, I entitle the schema of this concept.

Kant, p. 182

The image of a concept is a very useful function of the process of understanding a notion in mathematics, but after the clarification in Kant's analysis of Schematism, I think we need another term as frame of Davis (Vienner 1992), or sketch as we mean a rough draft, a drawing of a concept or a proof. So I use next the Figures of Vienner by the meaning of sketch in case of image.

In the present article we do not attempt a full theory for the representations but we focus on indicative application on a concrete mathematical concept. We deal with the concept of infinite, that is challenging concept for the students (Finsbein & all, 1981, Tirosh, 1992, Tsamir, 1999, 2002) and we present some aspects of its meaning connected with the Cantor Set.

Discussing about infinity leads to many fascinating paradoxes. By closely examining these paradoxes we learn a great deal about human mind, its powers, and its limitations. The first paradox for the intuition is the property: every infinite set has a proper subset with the same cardinality. Next it is a fault idea in students minds that all infinite sets have the same number of elements (Tirrosh 1992).

In a more sophisticated discussion we can say that the concept of infinity is "transcendental, in the sense that it transcends, or goes beyond, the physical limitations of any organism" Lakoff (1987, p xi). In the book Lakof & Núñez (2000, p. 158) tried solve the "mystery" of infinity by a reduction to the Basic Metaphor of Infinity (BMI). They advocate for the main argument of Aristotle about potential infinity, who distinguishes it from actual infinity, which is the infinity conceptualised as a realised "Thing". "The idea of actual infinity in mathematics is metaphorical, that the various instances of actual infinity make use of the ultimate metaphorical result of a process without end".
The Cantor set is well known in the Theory of Sets and very useful in Topology, Analysis or Theory of Fractals for the construction of counter-examples with funny properties as are Cantor fan, the indecomposable continuum of Brouwer, biconnected of Knaster-Kuratowski, (see in Lelek 1964, Engelking - Sieklucki 1992, Steen - Seebach 1978, Samuels - Samuels 1999). It is a very simple fractal gives us the possibility to expose in the intuition a comparison of the two kinds of mathematical infinity: The countable (with cardinality of natural numbers) and the uncountable (with cardinality of real numbers). The comparison of infinite sets is an obstacle for the students (Fischbein and all, 1979, Tirosh, 1992, Tsamir 1999).

The concept of Cantor set is available by several alternative representational modes geometric, symbolic and sketch mode. Every way has some advantage and ascribes a meaning into another context.

§. 1. CANTOR SET

Symbolic definition: The Cantor discontinuum is the $\mathbb{C}$ of all numbers $t$ of the form:

$$t = t_1/3 + t_2/3^2 + ... + t_n/3^n + ...,$$

where $t_n$ assumes one of the values 0 or 2. They are therefore numbers of the interval $[0, 1]$ that can be written in the ternary system of calculation without using digit 1.

For example $1/3$ belongs to $\mathbb{C}$ because

$$1/3 = 0/3 + 2/9 + 2/27 + ... + 2/3^n + ... = (0.0222...)_3$$

but $1/2$ does not belong to $\mathbb{C}$.

This definition follows the description by Vienner in Figure 1 and left many things implicit to students. The symbolic concepts have problems in understanding. Vienner believes that:

… to acquire a concept means to form a concept image for it. To know by heart a concept definition does not guarantee understanding of the concept. To understand, so we believe means to have a concept image.

(Vienner 1992, p 69).
Except the symbolic representation of Cantor set is given by an equivalent geometric definition as we see below. Every different representation, symbolic or geometric gives another aspect of the core concept and unfolds another feature of this. So we can do implicit properties that are hidden giving a global understanding. Vergnaud advocating the significance of representation argues by the position:

Figure 1. Purely formal deduction by Vienner

Most of these concepts remain totally implicit, but they may also be explicit, or become explicit: It is one of the aims of mathematical education to build up explicit and general concepts and theorems from such local intuitions.

Vergnaud (1998, p. 171)

The concept sketch (image) as is dealt by Tall & Vienner (1981) should be connect not only to definition, but with the progress of the proofs or other relationships we develop using the definition.

**Geometrical construction of Cantor set:** In this stage the teacher should do as a reminder how a number has a position in the closed interval $[0, 1]$ and the geometrical meaning of the series:

$$d = \sum_{n=1}^{\infty} \frac{d_n}{10^n}$$

where $d_n$ assumes one of the values 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 in the decimal system of well known measure by metre.

$$t = \sum_{n=1}^{\infty} \frac{t_n}{3^n}$$
where \( t_n \) takes one of the values 0, 1 or 2, in the ternary system as given before and

\[
b = \sum_{n=0}^{\infty} b_n 2^n
\]

where \( b_n \) assumes one of the values 0 or 1 in the binary system that we need next in the step function. The geometrical presentation of the previous systems gives a better understanding of the Cantor set and the others function that are provided by using of the set.

Let \( I = [0,1] \) is the closed interval of the real numbers that is defined:

\[
F_0 = I = \{ x \in \mathbb{R} : 0 \leq x \leq 1 \}
\]

\( \mathbb{R} \) the set of real numbers. (Closed is a set \( A \) if contains all points that are limit of a sequence of points of \( A \subseteq \mathbb{R} \). In the case of interval contains its ends as well.

Let us divide the interval \( I \) into 3 equal parts and let us remove the middle open interval.

\[
\Delta_i = \{ x : \frac{1}{3} \leq x \leq \frac{2}{3} \}
\]

The remaining set is the union of two closed subintervals each of length \( \frac{1}{3} \),

\[
F_1 = \{ x : 0 \leq x \leq \frac{1}{3} \}, \{ x : \frac{2}{3} \leq x \leq 1 \}
\]

In every one of these two closed subintervals we do the same, dividing them in three equal parts and remove their (open) middle parts. (Figure 2 is the sketch of the procedure, stage \( F_i \)).

Now we have subtracted a subset \( \Delta_2 \) of the interval \( I \), that is the union of two disjoint open intervals of length \( \frac{1}{9} \) each:

\[
\Delta_2 = \{ x : \frac{1}{9} < x < \frac{2}{9} \} \cup \{ x : \frac{7}{9} < x < \frac{8}{9} \}
\]

Next after the subtraction of the sets \( \Delta_1, \Delta_2 \) from the interval \( I \), it rests a smaller set that is the union of four segments of length \( \frac{1}{9} \) each. These closed intervals are the set \( F_2 \) as union of sets:
We continue this procedure for each one of these four intervals. We remove a set \( \Delta_3 \), that is the union of four disjoint open intervals with length \( \frac{1}{27} \) each one. The remaining set \( F_3 \) is the union 8 disjoint equal closed intervals.

Figure 2. Sketch of the first steps constructing the Cantor set

Continuing by this way in the \( n \)-step we remove a set \( \Delta_n \), which is the union \( 2^{n-1} \) disjoint open intervals each of length \( \frac{1}{3^n} \). After \( n \)-step from interval \( I \), it will remain the union of \( 2^n \) disjoint closed intervals each of length \( \frac{1}{3^n} \). Deleting from the interval \( I \) the union of the removed sets \( \Delta_n \) for each \( n = 1, 2, \ldots \), we obtain the set which defined previously arithmetically and we call Cantor set. (Obviously the sets \( F_n \) are closed and \( F_{n+1} \subset F_n \) for \( n = 1, 2, \ldots \)).

The \( C \) is defined by the equalities:

\[
C = I - \bigcup_{n=0}^{\infty} \Delta_n = \bigcap_{n=0}^{\infty} F_n
\]
**Basic properties of Cantor set:** The set $\Delta_n$ is the union of $2^{n-1}$ disjoint open intervals each of length $\frac{1}{3^n}$. The measure of $\Delta_n$ is $\frac{2^{n-1}}{3^n}$. Therefore is provided that the measure of the set $C$ is:

$$1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1 - \frac{\frac{1}{3}}{1 - \frac{2}{3}} = 1 - 1 = 0$$

so the Cantor set has *measure zero*.

Consequently the union:

$$\Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_n$$

with $n \to \infty$, fills the interval $I$ and its length tends to 1. So every interval $E$ belonging to interval $I$, should has common points with $\Delta_n$ for at least one $n = 1, 2, 3\ldots$:

$$E \cap \bigcup_{n=1}^{\infty} \Delta_n \neq \emptyset$$

and then we conclude that the interval $E$ cannot belong to the set $C$. Namely, Cantor set does not contain any interval (property of boundary set in the interval $I$, Kuratowski, p 211). Consequence of this is that the Cantor set is a *total discontinuum*.

Other properties are the *closed set* and *dense in itself* (it means a set each of whose point is an accumulation point of this set) and so it is *perfect* (closed and dense in itself)

**Note 1:** A student’s obstacle is observed, when we approach geometrically Cantor set, the intuition (Figure 3) is sometimes led to the fault to consider as elements of the set $C$ only the end points the excluded open intervals. These ends are countable, as we know the following theorem, (Kuratowski p. 70):

*The union of a countable sequence of countable sets is countable.*

Cantor set is uncountable as we show below. The symbolic treatment of the case act as a microscope and illuminates the fault of the intuition.
§. 2. CONTINUOUS MAPPINGS OF THE CANTOR DISCONTINUUM

Continuing we do a construction that shows the equivalence of cardinalities between Cantor discontinuum and the continuum of real numbers in the closed interval $[0, 1]$. The understanding of the process of the proof is more fruitful if we have in our mind a sketch of geometrical meaning of the representing numbers in ternary and binary systems, (Figure 4). The next proof is from Engelking & Sieklucki, (p, 200).

The step function (or staircase function): We construct a continuous map $f$ of the Cantor set $C$ onto unit interval $I$. If a real number $r \in C$ has the base 3 expansion:

$$r = \sum_{i=1}^{\infty} r_i 3^{-i}, \text{ with } r \neq 1, \text{ for } i = 1, 2, \ldots$$

then we take

$$f(r) = \frac{1}{2} \sum_{n=1}^{\infty} r_i 2^{-i}.$$  

We have of course $0 \leq f(r) \leq 1$, for each $r \in C$.

We prove the assertion that the function $f$ is uniformly continuous:

Let $r = \sum_{i=1}^{\infty} r_i 3^{-i}$, $r' = \sum_{i=1}^{\infty} r'_i 3^{-i}$ and let $\varepsilon > 0$. Choose a natural number $k$ so that $2^{-k} < \varepsilon$. If $|r - r'| < 3^{-k}$ then $r_i = r'_i$ for $i = 1, 2, \ldots, k$, we have:
\[ |f(r) - f(r')| \leq \frac{1}{2} \sum_{i=1}^{\infty} |r_i - r'_i| 2^{-i} \leq 2^{-k} < \varepsilon. \]

For every \( s \in I \) consider any expansion of \( s \) to base 2, say \( \sum_{i=1}^{\infty} s_i \cdot 2^{-i} \); thus \( s_i \) is 0 or 1, for \( i = 1, 2, \ldots \). Taking \( r_i = 2 s_i \) we have \( r_i = 0, 2 \) for \( i = 1, 2, \ldots \), so
\[
s = \sum_{i=1}^{\infty} s_i \cdot 2^{-i} = \frac{1}{2} \sum_{i=1}^{\infty} r_i \cdot 2^{-i} = f(r)
\]
where \( r = \sum_{i=1}^{\infty} r_i \cdot 3^{-i} \in C \). Thus \( f(C) = I \).

Figure 4. Interplay between definition and image by Vinner

Extension of \( f \): The function \( f : C \to I \) has a natural continuous extension \( f^* : I \to I \), which we describe bellow.

Notice that if \( r' \) is a left endpoint and \( r'' \) a right endpoint of an interval excluded at the \( n^{th} \) step of the construction of the Cantor set, then:
\[
r' = \sum_{i=1}^{n} r_i \cdot 3^{-i} + 2 \sum_{i=n+1}^{\infty} 3^{-i} \quad \text{and} \quad r'' = \sum_{i=1}^{n} r_i \cdot 3^{-i} + 2 \cdot 3^{-n}
\]
where \( r_i \neq 1 \) for \( i = 1, 2, \ldots, n-1 \). Then
\[
f(r') = \frac{1}{2} \sum_{i=1}^{n} r_i \cdot 2^{-i} + \sum_{i=n+1}^{\infty} 2^{-i} = \frac{1}{2} \sum_{i=1}^{n} r_i \cdot 2^{-i} + 2^{-n} = f(r'')
\]
that is \( f(r') = f(r'') \). Taking \( f^+(r) = f(r') + f(r'') \) for \( r' \leq r \leq r'' \), where \( r', r'' \) the endpoints of an exclude interval, in the progress of the construction of Cantor set, we obtain a continuous extension \( f^*: I \rightarrow I \) of the map \( f: C \rightarrow I \).

In view of the characteristic shape of its graph the function \( f^* \) is called step function, (Figure 5).

The same name is also given to the map \( f: f^*|C \). Seeing the step function we can realise that the Cantor set is uncountable.

Figure 5 Graph of step function

§. 3. AN ALTERNATIVE PRESENTATION OF CANTOR SET AND STEP FUNCTION

We have presented Cantor set symbolically by the ternary series. The number \( 1/3 \) of Cantor set by the series
Let us construct of a binary tree. I am using the description of Rucker in the Figure 6:

“The tree is constructed by letting a path move upward, forking infinitely means times. By continually halving distances, we fit in infinite forks below the horizontal line. We can imagine each point on this line as being a "leaf" that lies at the end of one of infinitely zigzagged branches up through the tree. A branch that goes all the way up through the tree is given by an infinity (countable) sequence of binary decisions. If one takes a pencil and traces a branch up through the tree as drawn, then one can describe one's path by an infinite sequence such that <Left, Right, Right, Left, Right,...>. Now it is clear that if we replace "Left" by “0” and "Right" by “1”, then each path can be identified with a member of \( 2^\mathbb{N} \) with the set of all branches up through the binary tree”.

Rucker (1983, p 259)

It is the set of all binary infinite sequences and well known theorem in set theory that:

\[ 2^{\mathbb{N}} = \mathfrak{c} \]

Here we can use as \( N \) the cardinality of natural numbers. Namely the infinite Cartesian product of the set \{0,1\} has the cardinality of real numbers (Kamke, Kuratowski)

We remark at first that the geometric construction of binary system of numbers in the interval of \([0,1]\) has the same effect as the previous one in the construction of binary tree. Divide the \( I \) in two and next the subintervals into two as well and so on. Further, we observe that an analogue action is attempted in geometric construction of Cantor set as we have in each level a Right or Left option set a number. This geometrical observation is enough show by a geometrical way the idea of step function.

Step function assigns to each number \( t \in I \) that has in ternary a development:

\[ t = (t_1, t_2, ...)_3 \]

where \( t_n = 0, 1 \) or \( 2 \) for \( n = 1,2,..., \) a number \( b \in I \) with the development in binary system:

\[ b = (b_1, b_2, ...)_2 \]

by the replace in ternary development each \( 2 \) by \( 1 \).

Therefore we realise that the function,

\[ s : \mathbb{C} \rightarrow I \]
is continuous because: If two elements of Cantor set are "near", then their images under the functions are "near" as well, because the nearer the two elements of the Cantor set, are the bigger part of their ternary representation is identified.

This mapping is on but is not 1-1. (Hint, In binary system the number 0,0111... is congruent with 0,1000..., as in decimal the number 0,0999... with 0,1000...).

Note 2: It is obvious that the ends of removing open intervals during the geometrical construction of Cantor set are those $t$ for which $\exists n_0 \in \mathbb{N} \exists$ such that $t_n = 0$ or $t_n = 2$ only, $\forall n > n_0$, $N$ natural numbers.

Figure 6: infinite binary tree

**CONCLUSION**

The challenges with the concept of infinity are presented to students. The particular idea of infinity encompasses many metaphors that transcend the experience as well the intuition. The metaphors are hidden inside the symbolic forms and comprehensive formulations. The
difficulties depend on the specific theoretical context that we evoke the meaning of infinity, Calculus, Set Theory, Topology, Algebra.

The Cantor set is available to alternative representations two are symbolic and another geometric. Each of them gives an alternative possibility for the treatment of the notion of infinity. Therefore we have three different stages of understanding in the sketches as seen in figures 2, 5 and 6, in geometric model, and two different symbolic representations. It is a key example that offers a combination of symbolic and intuitive representations and we can show the values and the limits each of them.

The inquiring of representations that involve the infinity and exposes it in a possible intuition help us in a global and more coherent acquisition of the infinity's concepts. Naturally when we evoke the intuition we should be careful of potential misleading. We need the continuous control by the functions of symbols and the inverse. So we need “to develop reflective abilities and critical thinking” Vienner (2002, p 9). between these two poles of intuition and symbolic manipulation. The symbols guarantee a precise knowledge because of clear rules of manipulations but the intuitive and visual ways helps for the spatial inserts of the concepts and proofs (Lakoff – Núñez, 2000).

The Cantor set with its paradox gives us the comparison of the two kinds of mathematical infinity, countable and non-countable. In the Cantor's diagonalization proof we apply 4 times BMI principle (Lakoff – Núñez, p 210). With the theorem we have the proof of uncountability of real numbers but we cannot uncover the difference with countable because we have no any comparison unfolded by our intuition. The Cantor set and step function provided us the this possibility. In addition Cantor set is the tool for the construction a lot of fascinate counter-examples.

REFERENCES


Kant I. (1929), Critique of Pure Reason, MACMILLAN
Kuratowski K. (1977), Introduction to Set Theory and Topology, PWN.
Tsamir P. (2002), From Primary to Secondary Intuition, Prospective Teachers’ Transitory Intuitions of Infinity, Mediterranean Journal in Mathematics Education (1), 1 , p 11-29.